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# A VARIATIONAL APPROACH FOR VERTICAL DEFORMATION ANALYSIS OF PILE GROUP

W. Y. SHEN, Y. K. CHOW AND K. Y. YONG

*Centre for Soft Engineering, National University of Singapore, Singapore*

## SUMMARY

A variational approach for the analysis of vertical deformation of pile groups is presented. The method assumes that the deformation of piles can be represented by a finite series. The method applies the principle of minimum potential energy to determine the deformation of piles. Using this method, an analytical solution for pile groups in soil modelled by the theoretical load–transfer curves can be obtained rigorously. Analysis of field tests indicates that the method can predict the performance of pile groups reasonably well.

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## 1. INTRODUCTION

Many analytical and numerical approaches have been applied to analyse vertically loaded pile groups. Among these, the integral equation method[1,2] (also known as the boundary element method) is the most well known. The method has also been extended to consider a certain class of soil inhomogeneity.[3,4] There are still some limitations for the method when applied to general soil inhomogeneity problems.

Randolph and Wroth[5] have presented a closed-form solution for single piles. They[6] have also extended the approach to develop an approximate analytical method of pile groups. In their method, theoretical load–transfer curves are used to model soil load-displacement relationship. The solution procedure is rigorous only for rigid pile groups, but for compressible pile groups, the solution needs some approximate equations to relate the settlement of the pile tops to that of pile bases. Note that for a single pile in soil with stiffness increasing linearly with depth, the solution for a compressible pile is also approximate.

Chow[7] has presented a hybrid method for the analysis of vertical deformation of pile groups. In this method, analysis of individual piles within a pile group is carried out using theoretical load–transfer curves to describe soil response. Analysis of pile–soil–pile interaction is performed using Mindlin's solution. Chow[8] has also presented another numerical method in which analysis of individual piles and pile–soil–pile interaction are all based on the theoretical load–transfer curves.

* Correspondence to Y. K. Chow, Centre for Soft Ground Engineering, National University of Singapore, Singapore.

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This paper examines the application of a variational approach for the analysis of a general pile group. The method assumes that the displacement of piles can be represented analytically by a function which has a set of undetermined constants. The displacement of piles characterized by these constants can be determined by the principle of minimum potential energy.

In principle, the method can handle the various kinds of soil model, but as a first step the theoretical load–transfer method similar to that used by Randolph and Wroth is chosen to represent the soil response. In this case, nondiscretization of either soil or piles is need. An analytical solution for pile groups embedded in homogeneous soil or in soil with stiffness varying with depth can be obtained rigorously, and the load–settlement relationship of group piles at the pile heads can be determined directly.

## METHOD OF ANALYSIS

### *Variational formulation*

Following variational principle, for a general group of piles, the total potential energy can be expressed in the form

$$\pi_p = \sum_{i=1}^{np} \frac{1}{2}\int\int\int_V E_p\left(\frac{\partial w}{\partial z}\right)^2 \mathrm{d}v + \frac{1}{2}\int\int_S \{\tau\}^{\mathrm{T}}\{w\}\,\mathrm{d}s + \frac{1}{2}\int\int_A \{\sigma\}^{\mathrm{T}}\{w_b\}\,\mathrm{d}A - \{w_t\}^{\mathrm{T}}\{p_t\} \qquad (1)$$

In equation (1), $np$ denotes the total number of piles in the group, $V$ denotes the volume of each pile, $S$ is the surface area of each pile, $A$ is the cross-sectional area of each pile, $E_p$ is the Young's modulus of piles. $\{\tau\} = \{\tau_1\tau_2\,\ldots\,\tau_{np}\}^{\mathrm{T}}$ is the vector of shear stresses of each pile at the pile-soil interface at depth $z$ (see Figure 1). $\{w\} = \{w_1 w_2\,\ldots\,w_{np}\}^{\mathrm{T}}$ is the vector of displacements of each pile at depth $z$, $\{\sigma\} = \{\sigma_1\sigma_2\,\ldots\,\sigma_{np}\}^{\mathrm{T}}$ (see Figure 1) and $\{w_b\} = \{w_{b1}w_{b2}\,\ldots\,w_{bnp}\}^{\mathrm{T}}$ are the vectors

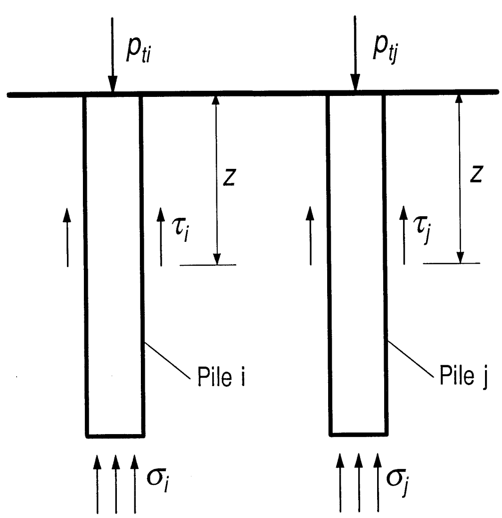


Figure 1. Stresses acting on pile $i$ and pile $j$

of normal stresses and displacements of each pile at the pile base, respectively, $\{p_t\} = \{p_{t1} p_{t2} \ldots p_{tnp}\}^{\mathrm{T}}$ and $\{w_t\} = \{w_{t1} w_{t2} \ldots w_{tnp}\}^{\mathrm{T}}$ are the vectors of external forces and displacements of each pile at the pile head, respectively.

The first component in equation (1) corresponds to the elastic strain energy of piles in the group, the second and third component corresponds to the work done by $\{\tau\}$ and $\{\sigma\}$, respectively, and the last component corresponds to the work done by $\{p_t\}$.

It is assumed that the deformations of piles and soil at the pile–soil interface are compatible, so the relationship between $\{\tau\}$ and $\{w\}$, $\{\sigma\}$ and $\{w_b\}$ depends on the soil models used. The soil models which have been widely used in this field are the load–transfer curve model (also known as the subgrade reaction model) and the elastic continuum model. Although equation (1) developed above is applicable either to the load–transfer curve model or elastic continuum model, only the load–transfer curve model is considered in this paper.

The load–transfer curve model can be further divided into two kinds: namely the well-known empirical load–transfer curves proposed by Coyle and Reese[9] and the theoretical load–transfer curves proposed by Randolph and Wroth,[5,6] and Kraft *et al.*[10] The empirical load–transfer curves are limited only to the analysis of single piles, but the theoretical load–transfer model can also be used to analyse pile groups.[6]

When the theoretical load–transfer curves are used, the relationship between shear stress $\{\tau\}$ and displacement $\{w\}$ and normal stress $\{\sigma\}$ and displacement $\{w_t\}$ can be assumed to be given, respectively, by

$$\{\tau\} = [k]\{w\} \tag{2}$$

$$\{\sigma\} = [k_b]\{w_b\} \tag{3}$$

in which $[k]$ = soil stiffness matrix for pile shaft at depth $z$, $[k_b]$ = soil stiffness matrix for pile base.

Using equations (2) and (3), the total potential energy in equation (1) can be expressed as

$$\pi_p = \sum_{i=1}^{np} \frac{1}{2} \int\int\int_V E_p \left(\frac{\partial w}{\partial z}\right)^2 \mathrm{d}v + \frac{1}{2}\int\int_S \{w\}^{\mathrm{T}}[k]\{w\}\mathrm{d}s + \frac{1}{2}\int\int_A \{w_b\}^{\mathrm{T}}[k_b]\{w_b\}\,\mathrm{d}A - \{w_t\}^{\mathrm{T}}\{p_t\} \tag{4}$$

For an elastic system in equilibrium, the principle of minimum potential energy requires

$$\delta\pi_p = 0 \tag{5}$$

where $\delta\pi_p$ is the variation in the total potential energy. It can be shown in equation (4) that $\{w\}$, $\{w_b\}$ and $\{w_t\}$ are the keys to the solution of the associated problem. In order to apply the variational principle for the analysis of pile groups, it is necessary to present displacement functions for each pile in the group. According to the deformation characteristic of a pile, the displacements of a pile will decrease gradually along its length towards the pile base, so the following function is assumed to represent the displacement of each pile in the group

$$w_i(z) = \sum_{j=1}^{k} \beta_{ij}\left(1 - \frac{z}{l}\right)^{j-1} \quad (i = 1, 2, \ldots, np) \tag{6}$$

Equation (6) is a finite series, in which $l$ is the length of pile, $z$ is depth below pile head, and $\beta_{ij}$ are the unknown coefficients.

The principle of minimum potential energy requires that $\pi_p$ be an extremum with respect to the admissible displacement field characterized by $\beta_{ij}$. Hence,

$$\frac{\partial \pi_p}{\partial \beta_{ij}} = 0 \quad (i = 1, 2 \ldots np) \quad (j = 1, 2 \ldots k) \tag{7}$$

By making use of equation (7), the total potential energy functional described in equation (4) reduces to the form

$$\sum_{i=1}^{np} \iiint_V E_p \frac{\partial w_i}{\partial z} \left( \frac{\partial (\frac{\partial wi}{\partial z})}{\partial \beta_{ij}} \right) \mathrm{d}v + \iint_S \left\{ \frac{\partial w}{\partial \beta_{ij}} \right\}^{\mathrm{T}} [k]\{w\} \mathrm{d}s + \iint_A \left\{ \frac{\partial w_b}{\partial \beta_{ij}} \right\}^{\mathrm{T}} [k_b]\{w_b\} \, \mathrm{d}A = \left\{ \frac{\partial w_i}{\partial \beta_{ij}} \right\}^{\mathrm{T}} \{p_t\}$$

$$(i = 1, 2 \ldots np) \quad (j = 1, 2 \ldots k) \tag{8}$$

Equation (8) is the basic variational formulation for a general group of piles in soil modelled by the load–transfer curve model.

*Load-deformation relationship for soil*

The load–transfer curves describe the behaviour between mobilized shaft friction and pile deformation. In this paper, the theoretical load–transfer curves used by Randolph and Wroth[5, 6] is chosen to represent load–displacement relationship for soil. The vector of displacements of the group piles at depth $z$ along the pile shaft can be expressed as

$$\{w\} = \frac{1}{G_z} [F] \{\tau\} \tag{9}$$

in which $\{w\}$ and $\{\tau\}$ is the same in equation (1), $[F]$ is the flexibility matrix of soil for pile shaft. $G_z$ is the shear modulus of soil at depth $z$, $G_z$ can be an arbitrary function varying with depth $z$. The flexibility coefficients in $[F]$ are given by

$$f_{ij} = r_0 \ln\left(\frac{r_m}{r}\right) \quad (i = 1, 2 \ldots np) \quad (j = 1, 2 \ldots np) \tag{10}$$

in which $r = r_0$ for $i = j$ and $r = s_{ij}$ for $i \neq j$, $r_0$ is the pile radius, $r_m$ is an empirical radius at which shear stress in the soil becomes negligible, and $s_{ij}$ is the spacing between pile $i$ and pile $j$. Following the suggestion by Randolph and Wroth, $r_m$ is given by

$$r_m = 2{\cdot}5 l \rho (1 - v) \tag{11}$$

in which $\rho$ is inhomogeneity factor, and $v$ is the Poisson's ratio of the soil. If the soil modulus is proportional to depth, $\rho$ is the ratio of soil modulus at pile mid-depth to that at the pile base. The vector of displacements of group piles at the pile base can be expressed as

$$\{w_b\} = \frac{1}{G_l} [F_b] \{\sigma\} \tag{12}$$

in which $\{w_b\}$ and $\{\sigma\}$ is the same as in equation (2), $[F_b]$ is the flexibility matrix of soil for the pile base, $G_l$ is the shear modulus of the soil at the pile base, and the flexibility coefficient in $[F_b]$ for

 

$i = j$ is given by

$$f_{bii} = \frac{1 - \nu}{4r_0} A \quad (i = 1, 2 \ldots np) \tag{13}$$

and the flexibility coefficients in $[F_b]$ for $i \neq j$ are given by

$$f_{bij} = \frac{1 - \nu}{2\pi(s_{ij})} A \quad (i = 1, 2 \ldots np) \quad (j = 1, 2 \ldots np) \tag{14}$$

In the equations above, $A$ is the pile cross-sectional area. By inverting the flexibility matrix $[F]$ and $[F_b]$ and letting $[k_{ss}] = [F]^{-1}$ and $[k_{bb}] = [F_b]^{-1}$, we get the soil stiffness matrix defined in the equations (2) and (3) as

$$[k] = G_z [k_{ss}] \tag{15}$$

$$[k_b] = G_l [k_{bb}] \tag{16}$$

*Solution for piles in homogeneous soil or in soil with modulus varying linearly with depth*

Substituting the above load-deformation relationship of the soil into equation (8), the result obtained can be expressed in matrix notation as

$$([k_p] + [k_s][A] + [k_{sb}])\{\beta\} = \{p\} \tag{17}$$

in which $\{\beta\} = \{\beta_{11}\beta_{12} \ldots \beta_{1k}\beta_{21}\beta_{22} \ldots \beta_{2k} \ldots \beta_{np1}\beta_{np2} \ldots \beta_{npk}\}^{\mathrm{T}}$

$$\{p\} = \{p_{t1}p_{t1} \cdots p_{t1}p_{t2}p_{t2} \cdots p_{t2} \cdots \mathrm{p}_{tnp}p_{tnp} \cdots p_{tnp}\}^{\mathrm{T}}$$

$[k_p]$, $[k_s]$, $[A]$ and $[k_{sb}]$ are matrices of order $(np \times k) \times (np \times k)$. Their corresponding coefficients are given in Appendix 1.

Equation (17) can be expressed as

$$[h]\{\beta\} = \{p\} \tag{18}$$

in which $[h] = [k_p] + [k_s][A] + [k_{sb}]$.

Equation (18) can be inverted to give the following form:

$$[f]\{p\} = \{\beta\} \tag{19}$$

in which $[f] = [h]^{-1}$

According to the assumed deformation function in equation (6), the settlement of each pile at the pile top can be expressed as

$$w_{ti} = \sum_{j=1}^{k} \beta_{ij} \tag{20}$$

Based on equations (19) and (20), after the addition of rows and columns of matrix $[f]$ is made, the load–deformation relationship of group piles at the pile heads is

$$\{w_t\} = [f_t]\{p_t\} \tag{21}$$

 

in which $[f_t]$ is the flexibility matrix of a pile group at the pile heads. By inverting $[f_t]$, we get the expression

$$\{p_t\} = [k_t]\{w_t\} \tag{22}$$

in which $[k_t] = [f_t]^{-1}$. So far a single stiffness matrix which relates the loads and displacements at the pile heads of group piles and which considers the reinforcing effect of all the group piles has been developed. It has an important function and particular advantage in the general analysis of pile groups loaded through a pile cap of finite stiffness. For the special case of a rigid pile cap loaded by a non-eccentric load, this will result in a uniform deformation of the group piles. The load acting at the pile heads can be solved directly by equation (22). Substituting the load obtained into equation (19), we can get the $(np \times k)$ constants $\beta_{ij}$. The deformation of each pile at any depth can then be determined analytically.

## NUMERICAL RESULTS

In order to verify the accuracy of the approach described in the paper, firstly a comparison with the solutions reported by Randolph and Wroth[5,6] is made. The results shown in Tables I and II are for a single pile in homogeneous soil and in soil with modulus increasing proportionally with depth, respectively, and the results shown in Table III are for symmetric rigid pile groups in homogeneous soil. It is pointed out that the assumed finite series for pile deformation converge quickly, the results correspond to the term number $k = 4$.

It is shown that the load–settlement ratio $P_t/Gr_0w_t$ obtained by the present method for single piles and symmetric rigid piles groups in homogeneous soil is exactly the same as those obtained by Randolph and Wroth's method, and the $P_t/Gr_0w_t$ obtained by the two methods show some differences for single piles in soil with modulus increasing proportionally with depth, especially for long compressible piles. It is shown that agreement between the two methods for $w_t/w_b$ (the ratio of pile head settlement to pile base settlement) for single piles is also good.

In principle, the accuracy of the present method depends on the choice of the deformation function of group piles. The above results demonstrate that the deformation of piles can be

Table I. Comparison of results for a single pile in homogeneous soil

| Pile stiffness ratio $\lambda = E_p/G$ | | | 1,000,000 | 10,000 | 3000 | 100 | 300 |
|---|---|---|---|---|---|---|---|
| $l/r_0 = 40$ | $w_t/w_b$ | (1) | 1·00 | 1·05 | 1·16 | 1·51 | 3·00 |
| | | (2) | 1·00 | 1·05 | 1·13 | 1·42 | 2·61 |
| | $P_t/Gr_0w_t$ | (1) | 68·0 | 66·0 | 61·6 | 52·3 | 36·0 |
| | | (2) | 68·0 | 66·0 | 61·6 | 52·3 | 36·0 |
| $l/r_0 = 40$ | $w_t/w_b$ | (1) | 1·00 | 1·15 | 1·54 | 2·91 | 11·9 |
| | | (2) | 1·00 | 1·14 | 1·48 | 2·66 | 9·92 |
| | $P_t/Gr_0w_t$ | (1) | 111·5 | 101·6 | 84·9 | 60·4 | 35·0 |
| | | (2) | 111·5 | 101·6 | 84·9 | 60·4 | 35·0 |

Note: (1) present method, (2) Randolph and Wroth's method, $\nu = 0{\cdot}4$.

 

Table II. Comparison of results for a single pile in non-homogeneous soil

| Pile stiffness ratio $\lambda = E_p/G_l$ | | | | 1,000,000 | 10,000 | 3000 | 100 | 300 |
|---|---|---|---|---|---|---|---|---|
| $l/r_0 = 80$ | $P_t/G_l r_0 w_t$ | $\rho = 0{\cdot}7$ | (1) | 53·7 | 52·1 | 48·7 | 41·1 | 28·2 |
| | | | (2) | 53·7 | 52·0 | 48·5 | 41·0 | 28·3 |
| | | $\rho = 0{\cdot}5$ | (1) | 43·6 | 42·3 | 39·4 | 33·2 | 21·9 |
| | | | (2) | 43·6 | 42·2 | 39·2 | 33·0 | 22·9 |
| $l/r_0 = 80$ | $P_t/G_l r_0 w_t$ | $\rho = 0{\cdot}7$ | (1) | 86·0 | 78·6 | 65·9 | 46·7 | 26·2 |
| | | | (2) | 86·0 | 78·0 | 64·8 | 46·1 | 27·2 |
| | | $\rho = 0{\cdot}5$ | (1) | 68·0 | 62·0 | 51·7 | 35·6 | 18·4 |
| | | | (2) | 68·0 | 61·4 | 50·8 | 36·1 | 21·8 |

Note: (1) present method, (2) Randolph and Wroth's method, $v = 0{\cdot}4$.

Table III. Comparison of results for symmetrical rigid pile groups in homogeneous soil

| The number of group piles | | | 2 piles | 3 piles | 4 piles |
|---|---|---|---|---|---|
| $l/r_0 = 40$ | $P_t/G_l r_0 w_t$ | (1) | 44·1 | 33·0 | 27·4 |
| | | (2) | 44·1 | 33·0 | 27·4 |
| $l/r_0 = 80$ | $P_t/G_l r_0 w_t$ | (1) | 69·0 | 50·4 | 40·9 |
| | | (2) | 69·0 | 50·4 | 40·9 |

Note: (1) present method, (2) Randolph and Wroth's method, $v = 0{\cdot}4$.

represented accurately by the deformation functions proposed in the paper. It can be concluded that the variational method can be used to analyse the behaviour of pile groups rigorously, especially for the load–settlement ratio which is an important parameter for the analysis of pile groups.

In Figures 2–4, the results obtained from the present method for symmetric compressible pile groups and for a $3 \times 3$ pile group at a pile spacing of $s = 5r_0$ are compared to the corresponding results of the integral equation method obtained by Butterfield and Banerjee.[2] The agreement in the solutions for pile groups is reasonably good. These results compared with the integral equation method are similar to those obtained by Randolph and Wroth who have also compared their solutions with Butterfield and Banerjee's solutions. Because the load-settlement relationship for soil used in the analysis is based on the theoretical load–transfer curves, which ignore the interaction of soil between different positions along a pile, the load–settlement ratios of piles obtained will be greater than those given by integral equation analysis, especially for a general pile group. If the theoretical load–transfer curves are used to describe the load–deformation relationship of soil using the variational method developed in the paper, no discretization for either soil or piles is needed. This will simplify the work needed for the analysis of piles. On the other hand, in a real soil, the non-linear nature of soil will lead to less interaction between piles, so the present method may be used to estimate the response of piles for practical purposes.

 

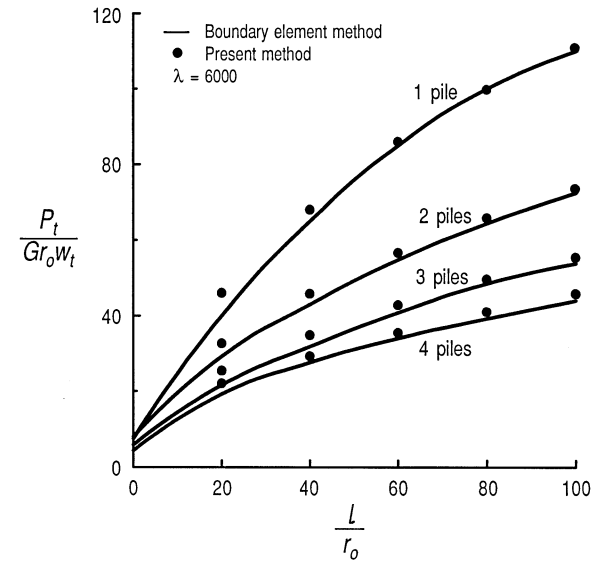


Figure 2. Load-settlement ratio for symmetric pile groups

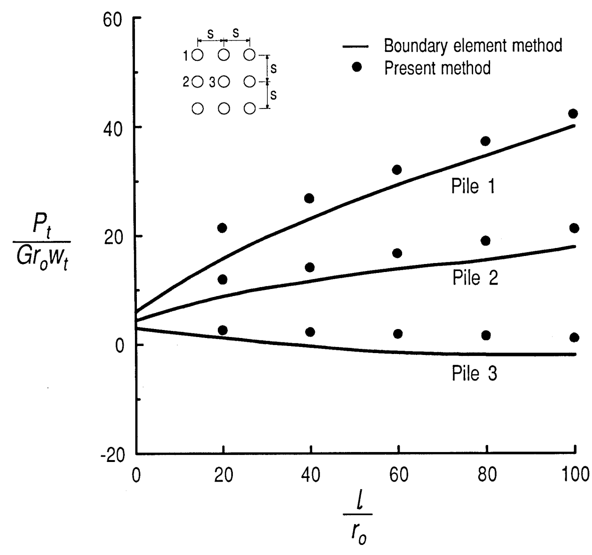


Figure 3. Load-settlement ratio for rigid $3 \times 3$ pile groups

## COMPARISON WITH FIELD TESTS

### *Jacked piles in London clay*

Cooke *et al.*[11] have reported the results of a series of field tests on pile groups embedded in London clay. The piles were tubular steel piles with external radius 84 mm and a wall thickness of

 

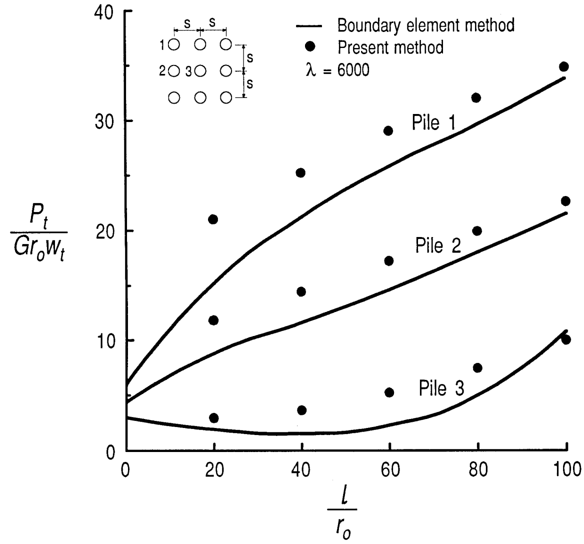


Figure 4. Load-settlement ratio for compressible $3\times3$ pile groups

6·4 mm. The piles were embedded to a depth 4·5 m with a spacing of three pile diameters. A series of load tests on a single pile and on a row of three piles (in the order A, B and C with pile A at the centre) was carried out.

The London clay in the test site extends from the surface to a depth of about 30 m. Its undrained shear strength $c_u$ based on the results given by laboratory triaxial tests increases linearly with depth. The value of the shear strength $c_w$ is about 32 kN/m$^2$ at the ground surface and about 78 kN/m$^2$ at the pile toe.

In order to analyse the field tests, it is assumed that the soil shear modulus is proportional to the undrained shear strength $c_u$, and that the soil Poisson's ratio $v = 0{\cdot}5$. Based on the assumption, a soil shear modulus profile with 15·6 MN/m$^2$ at the surface and 38·0 MN/m$^2$ at the base of the pile is deduced from back-analysis of pile A when loaded alone. Then the present method is used to estimate the interaction behaviour of the piles.

The measured and computed load–settlement curves for pile A when loaded alone and when piles A, B and C are loaded together (for simulating the effect of a absolute flexible pile cap) are shown in Figure 5.

The measured and computed load–settlement curves for the pile group and pile A in which the group piles had equal settlement (for simulating the effect of a rigid pile cap) are shown in Figure 6. The comparison shows that the computed results agree reasonably well with the measured field results.

### *Test on tubular steel piles in soft clay*

Koizumi and Ito[12] reported the testing of an isolated and a $3\times3$ pile group at a spacing of $s = 6r_0$ in soft silky clay. The piles were steel pipe piles with an external radius of 150 mm, and wall thickness of 3·2 mm. They were driven to a depth of 5·55 m. It was reported that the soil is

 

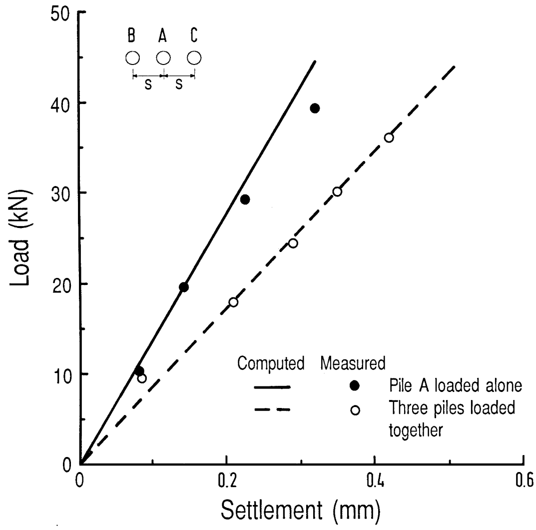


Figure 5. Load-settlement behaviour of pile A

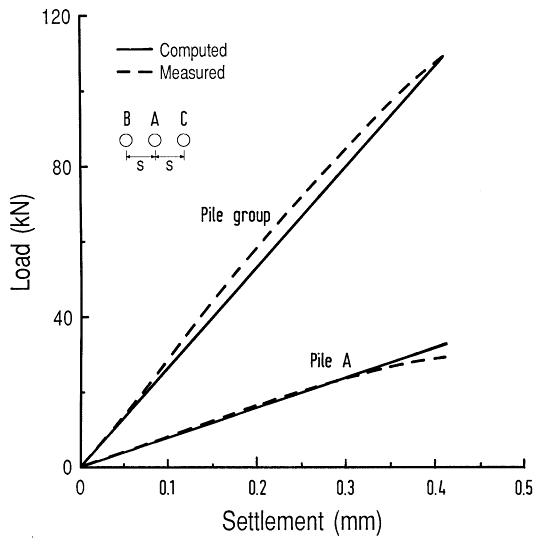


Figure 6. Load-settlement behaviour of piles

very sensitive, so after installation of the piles, it is reasonable to assume that the soil stiffness increases proportionally with depth. In the analysis, it is assumed that the soil inhomogeneity factor $\rho$ equals 0·65. A soil modulus profile with 1·7 MN/m$^2$ at the surface and 5·8 MN/m$^2$ at the base of the pile is deduced by back-analysis of the results from a single pile test under a working

 

Table IV. Comparison of settlement and load distribution in a $3\times 3$ pile group

| Pile settlement | | Pile load/average load | | |
|---|---|---|---|---|
| Computed | Measured | Pile position | Computed | Measured |
| 7.3 mm | 7.1 mm | Pile 1 | 1·29 | 1·25 |
| | | Pile 2 | 0·85 | 0·89 |
| | | Pile 3 | 0·40 | 0·46 |

Note: 1 corner pile, 2 Mid-side pile, 3 Centre pile.

load of $P = 150$ kN (about half the failure load). Then the settlement and load distribution of the $3\times 3$ pile group (under a working load of $P = 910$ kN) are predicted. The results are shown in Table IV. The agreement between computed results and measured field results is good.

The comparison above shows that if the soil parameters are determined from the back-analysis of a single pile, the performance of a pile group can then be estimated reasonably well.

## CONCLUSIONS

It is shown that the variational approach presented can be applied to analyse the vertical deformation of pile groups. It has been demonstrated that the deformation of group piles can be represented analytically by a finite series. For the soil modelled by theoretical load–transfer curves, rigorous analytical solutions for pile groups embedded in homogeneous soil or in soil with stiffness varying with depth can be obtained. The load-settlement relationship of group piles at the pile heads can be obtained directly with the present method. Analysis of field tests on pile groups shows that the variational method can estimate the performance of pile groups reasonably well.

Finally, it should be pointed out that the variational method presented in the paper is by no means restricted to the analysis of elastic problems. The method can be extended to consider non-linear behaviour of soils.

## APPENDIX

$[k_p]$, $[k_s]$, $[A]$ and $[k_{sb}]$ are matrices of order $(np\times k)\times(np\times k)$. They can be expressed as

$$[k_p] = \begin{bmatrix} k_{pp} & & & & \\ & k_{pp} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & k_{pp} \end{bmatrix} \quad [A] = \begin{bmatrix} A_s & & & & \\ & A_s & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & A_s \end{bmatrix}$$

$$[k_s] = \begin{bmatrix} k_{s11} & \cdot & \cdot & \cdot & k_{s1np} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ k_{snp1} & \cdot & \cdot & \cdot & k_{snpnp} \end{bmatrix} \qquad [k_{sb}] = \begin{bmatrix} k_{sb11} & \cdot & \cdot & \cdot & k_{sb1np} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ k_{sb1np} & \cdot & \cdot & \cdot & k_{bnpnp} \end{bmatrix}$$

The sub-matrices $[k_{pp}], [A_s]$ are of order $k \times k$; their corresponding coefficients are given by

$$k_{ppij} = E_p \frac{\pi r_0^2 (i-1)(j-1)}{l(i+j-3)}, \qquad A_{sij} = 2\pi r_0 l G_l \left( \frac{1}{i+j-1} + \frac{1-\alpha}{i+j} \right) \quad (i = 1, 2 \ldots k) \quad (j = 1, 2 \ldots k)$$

in which $\alpha$ is denoted as the ratio of soil modulus at ground surface to that at pile base. Homogeneous soil can be considered as a special case (i.e. $\alpha = 1$).

The sub-matrices $[k_{sij}], [k_{sbij}]$ are also of order $k \times k$; they can be expressed as

$$[k_{sij}] = \begin{bmatrix} k_{ssij} & & & & \\ & k_{ssij} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & k_{ssij} \end{bmatrix} \qquad [k_{sbij}] = \begin{bmatrix} k_{bbij} & & & & \\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & 0 \end{bmatrix}$$

$$(i = 1, 2 \ldots np) \quad (j = 1, 2 \ldots np)$$

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